Extended abstract

General guarantees for randomized benchmarking with random quantum circuits

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Assessing the quality of quantum gate implementations is crucial in developing quantum computers [1, 2]. Randomized benchmarking (RB) [3–8] and its many variants [9], such as linear cross-entropy benchmarking (XEB) [10], are arguably the most widely employed protocols for this purpose. RB measures the accuracy of random gate sequences of different lengths and reports the exponential decay rate of the resulting experimental signal with the sequence length. Stronger noise results in faster decays with smaller decay parameters.

Generally speaking, many experimental signatures can be rather well fitted by an exponential decay. Experimentally observing an exponential decay in an RB experiment does by itself not justify the interpretation of the decay parameter as a measure for the quality of the gates. In addition, RB requires a well-controlled theoretical model that explains the observed decays under realistic assumptions and provides the desired interpretation of the decay parameters.

RB is theoretically well-understood when the gates in the sequence are drawn *uniformly* at random from a compact group. In particular, generalizing the arguments of Refs. [8, 11–14], Helsen *et al.* [9] derived general guarantees for the signal form of the entire zoo of RB protocols with compact groups: If the noise of the gate implementation is sufficiently small (in a precise sense), each decay parameter is associated with an irreducible representation (irrep) of the group generated by the gates. To be precise, the decay parameter is the dominant eigenvalue of a generalized Fourier transform of the noisy implementation. Thus, the decay parameter indeed quantifies the average deviation of the gate implementation from their ideal action on the state subspace carrying the irrep.

In practice, however, the applicability of *uniform* RB protocols for holistically assessing the quality of noisy and intermediate-scale quantum (NISQ) hardware is limited. On currently available hardware, sufficiently long sequences of multi-qubit Clifford unitaries, for example, lead to way too fast decays to be accurately estimated for already moderate qubit counts. More scalable RB protocols *directly* draw sequences of local random gates, implementing a *random circuit* [15, 16]. We refer to those protocols that use a non-uniform probability distribution over a group as *non-uniform* RB protocols. The most prominent example of non-uniform RB is the *linear XEB* protocol that was used for the first demonstration of a quantum computational advantage in sampling tasks [10, 17].

Establishing theoretical guarantees for non-uniform RB is considerably more subtle. Roughly speaking, the interpretation of the decay parameter is more complicated as one additionally witnesses the convergence of the non-uniform distribution to the uniform one with the sequence length—causing a superimposed decay in the experimental data. These obstacles are well-known in the RB literature [16, 18] and have raised suspicion in the context of linear XEB [19, 20]. If not carefully considered, one easily ends up significantly overestimating the fidelity of the gate implementation.

The original theoretical analysis of linear XEB relies on the assumption that for every circuit, one observes an ideal implementation up to global depolarizing noise [10]. Building more trust in linear XEB has motivated a line of theoretical research, introducing different heuristic estimators [19]. Moreover, analyzing the behaviour of different noise models in random circuits [21, 22] using mappings of random circuits to statistical models [23]. However, general guarantees that work under minimal plausible assumptions on the gate implementation and for random circuits generating different groups—akin to the framework [9] for uniform RB—are missing.

In this work, we close this gap by developing a general theory of *filtered randomized benchmarking* with *random quantum circuits* under arbitrary gate-dependent (Markovian and time-stationary) noise. Besides linear XEB, *filtered RB* [9] encompasses character benchmarking [24] and Pauli-noise tomography [25], as well as variants of simultaneous [26], and correlated [27] RB as additional examples. Filtered RB protocols deviate from standard RB by omitting the last gate that inverts the sequence and instead perform a computational basis measurement. This approach significantly simplifies the experimental procedure and is arguably a core requirement for experimentally scalable *non-uniform* RB. The inverse gate is calculated in the classical post-processing of the data. At this stage, the experimental data can be additionally filtered to show only specific decays (associated with an individual irrep) of potentially overlapping decays arising for smaller groups.

More precisely, we consider a d-dimensional quantum system and a compact group $G < \mathrm{U}(d)$ equipped with a probability measure ν. The group G acts on complex $d \times d$ matrices in the *reference representation* $\omega(g) = U_g(\,\cdot\,)U_g^\dagger$. Then, the filtered RB protocol prepares a fixed initial state ρ, applies independent and identically distributed gates $g_1, \dots, g_m \sim \nu$, and performs a measurement in a fixed basis $E_i := |i\rangle\langle i|$ for $i \in \{1, \dots, d\}$. Repetitions yield samples of the form $(i^{(l)}, g_1^{(l)}, \dots, g_m^{(l)})$. In the post-processing, for an irrep ω_λ of ω with dimension d_λ, we compute the empirical mean $\hat{F}_\lambda$ of a *filter function* $f_\lambda(i, g_1, \dots, g_m)$ evaluated on the obtained samples:

$$\hat{F}_\lambda(m) := \frac{1}{N}\sum_{l=1}^{N} f_\lambda(i^{(l)}, g_1^{(l)}, \dots, g_m^{(l)}), \qquad f_\lambda(i, g_1, \dots, g_m) := \mathrm{tr}\left[E_i\, \omega(g_1 \cdots g_m) S^+ P_\lambda(\rho)\right] .$$

Here, P_λ is the projector onto the irrep ω_λ and S^+ is the pseudo-inverse of a frame operator associated with G and the measurement.

The filtering allows for a more fine-grained treatment of the argument at the heart of the framework of Ref. [9], and an individual analysis for the different irreps of the group G. In this way, we derive new perturbative bounds based on the harmonic analysis of compact groups that can be naturally combined with results from the theory of random quantum circuits, thereby treating uniform and non-uniform RB on the same footing.

Technically, we model the imperfect implementation of gates by introducing an *implementation map* ϕ on G such that $\phi(g)$ is completely positive and trace non-increasing for all $g \in G$. In the absence of noise, the implementation map should be exactly given by the *reference representation* ω of G. We then consider the operator-valued Fourier transform on the compact group G, which, when applied to ϕ *and the measure* ν, reads $\widehat{\phi\nu}[\omega_\lambda] = \int_G \omega_\lambda(g)^\dagger(\,\cdot\,)\phi(g)\,\mathrm{d}\nu(g)$. We treat this as a perturbation of the Fourier transform $\widehat{\omega\nu}[\omega_\lambda]$ for the ideal implementation $\phi = \omega$. Crucially, $\widehat{\omega\nu}[\omega_\lambda]$ has the form of a (second-order) *moment operator* associated with the measure ν, and the convergence rate of the random circuit generated by ν to the uniform measure on G is controlled by the spectral gap Δ_λ of $\widehat{\omega\nu}[\omega_\lambda]$. Using matrix perturbation theory, we then find an explicit expression for the data form of filtered RB:

Theorem (Data form of filtered RB, informal)**.** *Suppose there is a* $\delta_\lambda > 0$ *such that*

$$\left\|\widehat{\phi\nu}[\omega_\lambda] - \widehat{\omega\nu}[\omega_\lambda]\right\|_\infty \leq \delta_\lambda < \frac{\Delta_\lambda}{5} .$$

Then, $\mathbb{E}[\hat{F}_\lambda(m)] = \mathrm{tr}\,(A_\lambda I_\lambda^m) + \mathrm{tr}\,(B_\lambda O_\lambda^m)$ *where the matrix* I_λ *(of the size of the multiplicity of* λ*) captures the gate noise, independent of SPAM, and the second term is suppressed as*

$$\left|\mathrm{tr}\,(B_\lambda O_\lambda^m)\right| \leq c_\lambda\,(1 - \Delta_\lambda + 2\delta_\lambda)^m , \tag{1}$$

with a constant c_λ *depending on the measurement and SPAM. Typically, we have* $c_\lambda = O(d_\lambda)$*.*

Our guarantee, thus, assumes that the *error of the average implementation (per irrep)* of only the gates actually appearing in the random circuit is sufficiently small compared to a irrep-specific spectral gap of the random circuit. Then, the data of filtered RB is well-described by an exponential decay $\mathrm{tr}[A_\lambda I_\lambda^m]$ if the circuit is sufficiently deep to suppress the second summand, and the largest eigenvalue of I_λ is the dominant contribution. By Eq. (1), the subdominant terms essentially reflect the mixing process of the random circuit with convergence rate $1 - \Delta_\lambda$. Concretely, we find that the following sequence length is typically sufficient to suppress the subdominant terms by ϵ:

$$m \geq 2\Delta_\lambda^{-1}\big(\log(d_\lambda) + \log(1/\epsilon) + O(1)\big) . \tag{2}$$

We evaluate this bound using results on random quantum circuits to arrive at concrete scalings of the circuit depth for specific examples [28–32]. Explicitly, we discuss the cases of local random circuit or brickwork circuit with Haar-random unitary gates and Clifford generators. We find for practically relevant examples that our result implies that a *linear* circuit depth in the number of qubits suffices for filtered RB.

Omitting the inverse gate comes at the price that the simple arguments for the sample-efficiency of standard RB do no longer apply to filtered RB. As in state shadow tomography [33], the post-processing introduces estimators that are, in general, only bounded exponentially in the number of qubits. Thus, the precise convergence of estimators calculated from polynomially many samples is a priori far from clear.

Generalizing our perturbative analysis of filtered RB data to higher moments, we derive general expressions for the sample complexity of filtered RB in terms of the corresponding moments of the noise-free and uniformly random implementation. Again, subdominant terms appear that, however, become negligible if the sequence length m is chosen large enough. In the worst case, this can mean a constant overhead compared to Eq. (2), but in many relevant cases the previous bound is already sufficient. Denote by $\mathbb{E}[f_\lambda^2]_{\mathrm{ideal}}$ the second moment of the filter function when the implementation is noise-free and the gates are drawn uniformly from the group. We prove the following statement.

Theorem (Sampling complexity of filtered RB, informal)**.** *Choose the sequence length m such that the subdominant terms are bounded by κ. If the number of samples fulfills $N \geq (\mathbb{E}[f_\lambda^2]_{\mathrm{ideal}} + \kappa)\varepsilon^{-2}\delta^{-1}$ then the mean estimator $\hat{F}_\lambda(m)$ is ϵ-precise with probability at least δ.*

Our result can be summarized as follows: *Under essentially the same assumptions that guarantee the data form of the protocol, filtered RB is at least as sample-efficient as bounds on the analogous protocol using noise-free, uniformly distributed unitaries.* Again important example protocols are already sample-efficient using linear circuit depth. Perhaps surprisingly, we find that filtered RB with single-qubit gates coming from a unitary 3-design (e.g. the Clifford group) has constant sampling complexity irrespective of the size of the non-trivial support of the irreps. This is in contrast to the related findings in state shadow estimation. Interestingly, a similar result in local dimension $q > 2$ does not hold. Filtered RB without entangling gates, a general variant of simultaneous RB, can, therefore, also be used to efficiently extract crosstalk measures between more than two-qubits in a simple experiment.

Finally, it is an open question whether the post-processing of filtered RB can be modified such that meaningful decay constants can already be extracted from *constant depth* circuits. In the context of linear XEB, Ref. [19] introduces a heuristic, so-called 'unbiased' estimator to this end that is also central to the proposal of Ref. [21]. Using the general perspective of filtered RB, we sketch two general approaches to construct a modified linear estimator for constant-depth circuits. The first approach introduces a more costly computational task in the classical post-processing. The second approach requires that the random distribution of circuits is locally invariant under local Clifford gates. We formally argue that these estimators work under the assumption of global depolarizing noise, putting them at least on the same footing as existing theoretical proposal, but leave a detailed perturbative analysis to future work.

We expect that our theory of non-uniform filtered RB can be applied to many more practically relevant benchmarking schemes and bootstraps the development of new RB schemes. In fact, one of our main motivations for deriving the flexible theoretical framework is its applications for the characterization and benchmarking of non-universal and analog quantum computing devices—consolidating and extending existing proposals [34, 35] in future work.

On a technical level, we develop tools to analyze noisy random circuits using harmonic analysis on compact groups and matrix perturbation theory. We expect that this perturbative description finds applications in quantum computing also beyond the randomized benchmarking of quantum gates. The tools and results might, in principle, be applicable to analyze the noise-robustness of any scheme involving random circuits, e.g. randomized compiling [36], shadow tomography and randomized measurements [37] or error mitigation [38]. As a by-product, our variance bounds take a more direct representation-theoretic approach working with tensor powers of the adjoint representation rather than exploiting vector space isomorphisms and invoking Schur-Weyl duality [33]. We hope that our approach also opens up a complimentary, illuminating perspective on the sample-efficiency of estimation protocols based on random sequences of gates more generally.

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