Quantum Information Methods for Many-Body Physics

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Exercise Sheet 3: Anticoncentration of Haar ensemble **Due: 21 May, 10:00**

Uniformly distributed states in the Hilbert space of N qudit are expected to be widely spread in any basis. This notion, denoted anticoncentration, is at the heart of several applications in quantum many-body systems, from conceptual problems to applications. In this exercise sheet, we will recover the moment of the overlap distribution $p_x = |\langle x|\Psi\rangle|^2$ for $\Psi = U|0\rangle$ random states generated by a Haar unitary distributed matrix $U \in \mathcal{U}(d^N)$. The probability of the overlaps p_x, formally defined by

$$\mathcal{P}(w) = \mathbb{E}_{x\in\mathcal{B}}\mathbb{E}_{U\in\mathcal{U}(d^N)}\left[\delta(w - |\langle x|\Psi\rangle|^2\right] \ ,$$

is a distribution of distribution and is given by the Porter-Thomas distribution. We will recover this from the knowledge of all the moments.

1 Expression of the moments (3 P)

Use the properties of the $\delta-$function to express the moments

$$I_k \equiv \int_{-\infty}^{\infty} dw\, w^k \mathcal{P}(w) \ , \tag{1}$$

in terms of the amplitude. Express this quantity as the trace $I_k = \mathrm{tr}(A\mathrm{M}_k(B))$: who are A and B?

2 First and second moment (2 P)

Compute the first moment I_1 and I_2 explicitly, recalling the Weingarten fundamental expression for generic $k \geq 1$

$$\mathrm{M}_k(B) = \sum_{\sigma,\tau\in\mathrm{S}_k} \mathrm{Wg}_{\sigma,\tau}(d) T_\sigma \mathrm{tr}(T_\tau^\dagger B) \ . \tag{2}$$

3 Generic moments (3 P)

Write the explicit value of I_k for generic moments. *Hint: from the previous exercise, you should have noticed already that the computation is easier then necessary because* $\mathrm{tr}(R_\sigma|0\rangle\langle 0|^k) = \spadesuit$ *a simple value for any permutation. What is* $\spadesuit$*?*

4 Generating function and Porter-Thomas distibution(3 P)

From the knowledge of all the moments, you can compute the generating function

$$F(z) = \sum_{k=0}^{\infty} (-z)^k \frac{I_k}{k!} \tag{3}$$

which is nothing but the Laplace transform of $P(w)$, namely $F(z) = \int dw P(w) e^{-zw}$. Use the inverse Laplace transform to obtain the $P(w)$.

5 [Optional] A less trivial computation (4 P)

Compute the expectation value

$$\Xi_k = \mathbb{E}[|\langle 1|U|0\rangle|^{2k}|\langle 0|U|0\rangle|^{2k}] \,, \tag{4}$$

For this: (i) write $\Xi_k = \mathrm{tr}(A\mathrm{M}_k(B))$ in terms of A and B to be determined. Who are these operators? (ii) Apply the Weingarten form of M_k, and reduce this to something like $\propto \sum_{\sigma\in S_k} \mathrm{tr}(T_\sigma A)$. (iii) Evaluate explicitly the final expression for $k=1$ and $k=2$. Can you guess the generic expression for any k?