Quantum Information Methods for Many-Body Physics

Xhek Turkeshi, Markus Heinrich

Exercise Sheet 2 **Due: 6 May, 12:00**

1 Traces of Operators and Permutations (4 P)

Consider the operator $\mathcal{O}_k \equiv (A \otimes B)^{\otimes k/2} = A \otimes B \otimes A \otimes B \otimes \cdots \otimes A \otimes B$, where k is even. Compute explicitly the value of $\langle \mathcal{O}_k \rangle_\pi \equiv \mathrm{Tr}(R_\pi \mathcal{O}_k)$, where $R_\pi \in \mathrm{S}_k$, for the following permutations

a) (0.5 P)

$$\pi = (12) \in \mathrm{S}_2.$$

b) (0.5 P)

$$\pi = (12)(35)(46) \in \mathrm{S}_6.$$

c) (0.5 P)

$$\pi = (1532)(46) \in \mathrm{S}_6.$$

d) (0.5 P)

$$\pi = (135)(426) \in \mathrm{S}_6.$$

e) (1 P)

$$\pi = (28)(16)(735)(4) \in \mathrm{S}_8.$$

f) (1 P)

$$\pi = (281)(6,7,3,4,5) \in \mathrm{S}_8.$$

Hint: The graphical notation can simplify the computation.

2 Gram Matrix and Weingarten Matrix (3+2 P)

The Gram matrix is defined by the overlap $\mathrm{G}_{\sigma,\pi} = \mathrm{Tr}(R_\sigma^\dagger R_\pi)$, where R_π are permutations operators acting on the replica space $\mathcal{H}^{\otimes k}$, with $\mathcal{H}$ of dimension d. For $d \geq k$, the Weingarten matrix Wg is the inverse of G, namely $\sum_{\pi \in \mathrm{S}_k} \mathrm{G}_{\rho,\pi} \mathrm{Wg}_{\pi,\sigma} = \delta_{\rho,\sigma}$.

- **a)** Compute the explicit form of the Gram matrix for S_k when: (a) $k = 1$ **(0.5 P)**, (b) $k = 2$ **(0.5 P)**, (c) $k = 3$ **(1 P)**.
- **b)** From the Gram matrix, compute the Weingarten matrix for S_k when: (a) $k = 1$ and $k = 2$ **(1. P)**, (c) (optional) $k = 3$ **(2 P)**.

Hint: use symbolic calculus software to help yourself invert G *for* $k = 3$.

3 Expectation Values on Symmetric Subspace (4+1 P)

Consider the symmetric projector $P_{\mathrm{sym},k,d}$ on the symmetric subspace $\mathrm{Sym}_{k,d}$ on k replica of an Hilbert space of dimension d. Compute the expectation values $\langle \mathcal{A}_k \rangle_{\mathrm{sym}} = \mathrm{Tr}(\mathcal{A}_k P_{\mathrm{sym},k,d})$ for the following operators $\mathcal{A}_k$ and replica number k

a) **(1 P)**

$$k = 2 \qquad \mathcal{A}_2 = (A \otimes B)^{\otimes k/2}$$

b) (Optional) **(1 P)**

$$k = 4 \qquad \mathcal{A}_4 = A^{\otimes 4}$$

c) **(1 P)**

$$\text{for generic } k \qquad \mathcal{A}_k = (|0\rangle\langle 0|)^{\otimes k}$$

d) **(1 P)**

$$k = 2 \text{ and } k = 4 \qquad \mathcal{A}_k = P^{\otimes k}$$

where P is an operator such that $P^\dagger = P$, $P^2 = \mathbb{1}$ and $\mathrm{Tr}(P) = 0$. (You can think of this as non-trivial Pauli strings).

e) **(1 P)**

$$\text{for } k = 2,\ k = 4 \text{ and } k = 6 \qquad \mathcal{A}_k = (|0\rangle\langle 0|)^{\otimes k/2} \otimes (|1\rangle\langle 1|)^{\otimes k/2}.$$